Exact solutions of the Gerdjikov-Ivanov equation using Darboux transformations

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February 2, 2015

Abstract

We study the Gerdjikov-Ivanov (GI) equation and present a standard Darboux transformation for it. The solution is given in terms of quasideterminants. Further, the parabolic, soliton and breather solutions of the GI equation are given as explicit examples.

Keywords: Gerdjikov-Ivanov equation; Derivative nonlinear Schrödinger equation; Darboux transformation; Quasideterminants.

2010 Mathematics Subject Classification: 35C08, 35Q55, 37K10, 37K35

1 Introduction

The well known nonlinear Schrödinger (NLS) equation is one of the most important soliton equations. Extended versions of this equation with higher order nonlinearity have been proposed and studied by various authors. Among them, there are three celebrated equations with derivative–type nonlinearities, which are called the derivative NLS equations (DNLS). One is the Kaup-Newell equation (DNLSI) [16]

$$iq_t + q_{xx} = i(|q|^2 q)_x,$$

the second is the Chen–Lee–Liu equation (DNLSII) [2]

$$iq_t + q_{xx} + i|q|^2 q_x = 0,$$

while the third is the Gerdjikov–Ivanov equation (DNLSIII) [11]

$$iq_t + q_{xx} + iq^2 q_x^* + \frac{1}{2} q^3 {q^*}^2 = 0, \tag{1.1}$$

where q^* denotes the complex conjugate of q. The NLS equation with its cousin the DNLS equations are completely integrable and play an important role in mathematical physics [1, 3, 14, 15, 17, 26].

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It is known that these three equations may be transformed into each other by a chain of gauge transformations and the method of gauge transformation can also be applied to some generalised cases [4, 18–20, 27]. Therefore, in principle, the corresponding results for Chen-Lee–Lue and the GI equations may be obtained from the Kaup-Newell equation. However, these transformations involve very complicated integrals and it is not easy to obtain their explicit forms. So, even though the three systems are related by gauge transformations it is more convenient to treat them each separately.

In [24], the explicit quasideterminant solutions of the Kaup Newell equation (DNLSI) are presented via a standard Darboux transformation. In this paper, we study the Gerdjokov-Ivanov equation (DNLSIII) to obtain explicit solutions by using a standard Darboux transformation. Darboux transformations are an important tool for studying the solutions of integrable systems. They provide a universal algorithmic procedure to derive explicit exact solutions of integrable systems. In recent years, there has been some interest in solutions of the Gerdjikov-Ivanov equation obtained by means of *Darboux-like* transformations [6, 12, 28]. These solutions are often written in terms of determinants with a complicated structure, where the determinant representations of n-fold Darboux transformations are obtained by stating and proving a sequence of theorems.

On the other hand, in the present paper, we present a systematic approach to the construction of solutions of (1.1) by means of a standard Darboux transformation and written in terms of quasideterminants [7, 8]. Quasideterminants have various nice properties which play important roles in constructing exact solutions of integrable systems [9, 10, 13, 23–25].

This paper is organized as follows. In Section 1.1 below, we give a brief review on quasideterminants. In Section 3, we state a standard Darboux theorem for the Gerdjikov-Ivanov system. In Sections 3.2 and 4, we present the quasideterminant solutions of the Gerdjikov-Ivanov equation by using the Darboux transformation. Here, the quasideterminants are written in terms of solutions of linear eigenvalue problems. In Section 5, particular solutions of the Gerdjikov-Ivanov equation are given for both zero and non-zero seed solutions. The conclusion is given in the final Section 6.

1.1 Quasideterminants

In this short section we recall some of the key elementary properties of quasideterminants. The reader is referred to the original papers [7, 8] for a more detailed and general treatment.

The notion of a quasideterminant was first introduced by Gelfand and Retakh in [7] as a straightforward way to define the determinant of a matrix with noncommutative entries. Many equivalent definitions of quasideterminants exist, one such being a recursive definition involving inverse minors. Let $A = (a_{ij})$ be an $n \times n$ matrix with entries over a usually non commutative ring

$$|A|_{ij} = a_{ij} - r_i^j \left(A^{ij}\right)^{-1} c_j^i, \tag{1.2}$$

where r_i^j represents the row vector obtained from i^{th} row of A with the j^{th} element removed, c_j^i represents the column vector obtained from j^{th} column of A with the i^{th} element removed and A^{ij} is the $(n-1) \times (n-1)$ submatrix obtained by deleting the i^{th} row and the j^{th} column from A. Quasideterminants can also be denoted by boxing the entry about which the expansion is made

$$|A|_{ij} = \begin{vmatrix} A^{ij} & c_j^i \\ r_i^j & \boxed{a_{ij}} \end{vmatrix}. \tag{1.3}$$

If A is an $n\times n$ matrix over a commutative ring, then the quasideterminant $|A|_{ij}$ reduces to a ratio of determinants

$$|A|_{ij}=(-1)^{i+j}\frac{\det A}{\det A^{ij}}. \tag{1.4}$$

It should be noted that the expansion formula (1.2) is also valid in the case of block matrices provided the matrix to be inverted is square.

In this paper, we will consider only quasideterminants that are expanded about a term in the last column, most usually the last entry. For example considering a block matrix $M=\begin{pmatrix} A & B \\ C & d\end{pmatrix}$, where A is an invertible square matrix over $\mathcal{R}$ of arbitrary size and B, C are column and row vectors over $\mathcal{R}$ of compatible lengths, respectively, and $d\in\mathcal{R}$, the quasideterminant of M is expanded about d is defined by

$$\begin{vmatrix} A & B \\ C & \boxed{d}\end{vmatrix}=d-CA^{-1}B. \tag{1.5}$$

2 Gerdjikov-Ivanov equations

Let us consider the pair of Gerdjikov-Ivanov equations

$$iq_t+q_{xx}+iq^2r_x+\frac{1}{2}q^3r^2 \;=\; 0, \tag{2.1}$$

$$ir_t-r_{xx}+ir^2q_x-\frac{1}{2}q^2r^3 \;=\; 0, \tag{2.2}$$

where $q=q(x,t)$ and $r=r(x,t)$ are complex valued functions. Equations (2.1) and (2.2) reduce to the Gerdjikov-Ivanov equation (1.1) for $r=q^*$ while the choice of $r=-q^*$ would lead to (1.1) with the sign of the nonlinear term reversed.

The Lax pair for the Gerjiov-Ivanov system (2.1)–(2.2) is given by

$$L \;=\; \partial_x+J\lambda^2-R\lambda+\frac{1}{2}qrJ \tag{2.3}$$

$$M \;=\; \partial_t+2J\lambda^4-2R\lambda^3+qrJ\lambda^2+U\lambda+W, \tag{2.4}$$

where J, R and U are 2×2 matrices such that

$$J=\begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix},\quad R=\begin{pmatrix} 0 & q \\ r & 0\end{pmatrix},\quad U=\begin{pmatrix} 0 & -iq_x \\ ir_x & 0\end{pmatrix} \tag{2.5}$$

and

$$W=\begin{pmatrix} -\frac{1}{2}\left(rq_x-qr_x\right)-\frac{1}{4}iq^2r^2 & 0 \\ 0 & \frac{1}{2}\left(rq_x-qr_x\right)+\frac{1}{4}iq^2r^2\end{pmatrix}. \tag{2.6}$$

Here λ is an arbitrary complex number called the eigenvalue (or spectral parameter).

3 Darboux Theorem and Dimensional Reductions

Theorem 3.1 ([5,21,22]). *Consider the linear operator*

$$L = \partial_x + \sum_{i=0}^{n} u_i \partial_y^i \tag{3.1}$$

where $u_i \in R$, where R is a ring, in general non-commutative. Let $G = \theta \partial_y \theta^{-1}$, where $\theta = \theta(x, y)$ is an invertible eigenfunction of L, so that $L(\theta) = 0$. Then

$$\tilde{L} = GLG^{-1} \tag{3.2}$$

has the same form as L:

$$\tilde{L} = \partial_x + \sum_{i=0}^{n} \tilde{u}_i \partial_y^i \tag{3.3}$$

If ϕ is any eigenfunction of L then

$$\tilde{\phi} = \phi_x - \theta_y \theta^{-1} \phi \tag{3.4}$$

is an eigenfunction of $\tilde{L}$. In other words, if $L(\phi) = 0$ then $\tilde{L}(\tilde{\phi}) = 0$ where $\tilde{\phi} = G(\phi)$.

3.1 Dimensional reduction of Darboux transformation

Here, we describe a reduction of the Darboux transformation from $(2 + 1)$ to $(1 + 1)$ dimensions. We choose to eliminate the y-dependence by employing a 'separation of variables' technique. The reader is referred to the paper [25] for a more detailed treatment. We make the ansatz

$$\begin{aligned} \phi &= \phi^r(x,t)e^{\lambda y}, \quad &(3.5)\\ \theta &= \theta^r(x,t)e^{\Lambda y}, \quad &(3.6)\end{aligned}$$

where λ is a constant scalar and Λ an $N \times N$ constant matrix and the superscript r denotes reduced functions, independent of y. Hence in the dimensional reduction we obtain $\partial_y^i(\phi) = \lambda^i \phi$ and $\partial_y^i(\theta) = \theta \Lambda^i$ and so the operator L and Darboux transformation G become

$$\begin{aligned} L^r &= \partial_x + \sum_{i=0}^{n} u_i \lambda^i, \quad &(3.7)\\ G^r &= \lambda - \theta^r \Lambda (\theta^r)^{-1}, \quad &(3.8)\end{aligned}$$

where θ^r is a matrix eigenfunction of L^r such that $L^r(\theta^r) = 0$, with λ replaced by the matrix Λ, that is,

$$\theta_x^r + \sum_{i=0}^{n} u_i \theta^r \Lambda^i = 0. \tag{3.9}$$

Below we omit the superscript r for ease of notation.

3.2 Iteration of reduced Darboux Transformations

In this section we shall consider iteration of the Darboux transformation and find closed form expressions for these in terms of quasideterminants.

Let L be an operator, form invariant under the reduced Darboux transformation $G = \lambda - \theta\Lambda\theta^{-1}$ discussed above.

Let $\phi = \phi(x,t)$ be a general eigenfunction of L such that $L(\phi) = 0$. Then

$$\begin{aligned}
\tilde{\phi} &= G_\theta(\phi) \\
&= \lambda\phi - \theta\Lambda\theta^{-1}\phi \\
&= \begin{vmatrix} \theta & \phi \\ \theta\Lambda & \boxed{\lambda\phi} \end{vmatrix}
\end{aligned}$$

is an eigenfunction of $\tilde{L} = G_\theta L G_\theta^{-1}$ so that $\tilde{L}(\tilde{\phi}) = \lambda\tilde{\phi}$. Let θ_i for $i = 1, \ldots, n$, be a particular set of invertible eigenfunctions of L so that $L(\theta_i) = 0$ for $\lambda = \Lambda_i$, and introduce the notation $\Theta = (\theta_1, \ldots, \theta_n)$. To apply the Darboux transformation a second time, let $\theta_{[1]} = \theta_1$ and $\phi_{[1]} = \phi$ be a general eigenfunction of $L_{[1]} = L$. Then $\phi_{[2]} = G_{\theta_{[1]}}\left(\phi_{[1]}\right)$ and $\theta_{[2]} = \phi_{[2]}|_{\phi\to\theta_2}$ are eigenfunctions for $L_{[2]} = G_{\theta_{[1]}} L_{[1]} G_{\theta_{[1]}}^{-1}$.

In general, for $n \geq 1$, we define the nth Darboux transform of ϕ by

$$\phi_{[n+1]} = \lambda\phi_{[n]} - \theta_{[n]}\Lambda_n\theta_{[n]}^{-1}\phi_{[n]}, \tag{3.10}$$

in which

$$\theta_{[k]} = \phi_{[k]}|_{\phi\to\theta_k}\ .$$

For example,

$$\begin{aligned}
\phi_{[2]} &= \lambda\phi - \theta_1\Lambda_1\theta_1^{-1}\phi = \begin{vmatrix} \theta_1 & \phi \\ \theta_1\Lambda_1 & \boxed{\lambda\phi} \end{vmatrix}, \\
\phi_{[3]} &= \lambda\phi_{[2]} - \theta_{[2]}\Lambda_2\theta_{[2]}^{-1}\phi_{[2]} \\
&= \begin{vmatrix} \theta_1 & \theta_2 & \phi \\ \theta_1\Lambda_1 & \theta_2\Lambda_2 & \lambda\phi \\ \theta_1\Lambda_1^2 & \theta_2\Lambda_2^2 & \boxed{\lambda^2\phi} \end{vmatrix}.
\end{aligned}$$

After n iterations, we get

$$\phi_{[n+1]} = \begin{vmatrix} \theta_1 & \theta_2 \ \ldots \ \theta_n & \phi \\ \theta_1\Lambda_1 & \theta_2\Lambda_2 \ldots \theta_n\Lambda_n & \lambda\phi \\ \theta_1\Lambda_1^2 & \theta_2\Lambda_2^2 \ldots \theta_n\Lambda_n^2 & \lambda^2\phi \\ \vdots & \vdots \ \ldots \ \vdots & \vdots \\ \theta_1\Lambda_1^n & \theta_2\Lambda_2^n \ldots \theta_n\Lambda_n^n & \boxed{\lambda^n\phi} \end{vmatrix}. \tag{3.11}$$

4 Constructing Solutions for Gerdjikov-Ivanov Equation

In this section we determine the specific effect of the Darboux transformation $G = \lambda - \theta\Lambda\theta^{-1}$ on the 2×2 Lax operators L, M given by (2.3),(2.4). Here θ is a eigenfunction satisfying $L(\theta) = M(\theta) = 0$

with 2×2 matrix eigenvalue Λ. By supposing that L is transformed to a new operator $\tilde{L}$, say, we calculate that the effect of the Darboux transformation $\tilde{L} = GLG^{-1}$ is such that

$$\tilde{R} = R - \left[J, \theta\Lambda\theta^{-1}\right] \tag{4.1}$$

and

$$\begin{aligned} \tilde{R}\theta\Lambda\theta^{-1} - \theta\Lambda\theta^{-1}R + \frac{1}{2}J\left(\tilde{q}\tilde{r} - qr\right) &= 0, & (4.2) \\ \left(\theta\Lambda\theta^{-1}\right)_x + \frac{1}{2}\left[J\left(\theta\Lambda\theta^{-1}\right)\tilde{q}\tilde{r} - \theta\Lambda\theta^{-1}Jqr\right] &= 0. & (4.3) \end{aligned}$$

From (4.2), we see that $\theta\Lambda\theta^{-1}$ must be an anti-diagonal matrix, antidiag(a, b), say, and then from (4.3) the multiplication of the anti-diagonal terms must be constant ($ab = constant$). Guided by this, we choose

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda. \tag{4.4}$$

Finally, the condition $\theta\Lambda\theta^{-1} = \text{antidiag}(a, b)$ leads to the requirement that the matrix θ has the structure

$$\theta = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}, \tag{4.5}$$

where $\theta_{11}\theta_{22} + \theta_{12}\theta_{21} = 0$.

For notational convenience, we introduce a 2×2 matrix $P = (p_{ij})$ $(i, j = 1, 2)$ such that $R = [J, P]$, and hence

$$P = \frac{1}{2i}\begin{pmatrix} p_{11} & q \\ -r & p_{22} \end{pmatrix}. \tag{4.6}$$

From (4.1), since $R = [J, P]$, we have

$$\tilde{P} = P - \theta\Lambda\theta^{-1} \tag{4.7}$$

which can be written in a quasideterminant structure as

$$\tilde{P} = P + \begin{vmatrix} \theta & I_2 \\ \theta\Lambda & \boxed{0_2} \end{vmatrix}. \tag{4.8}$$

We rewrite (4.7) as

$$P_{[2]} = P_{[1]} - \theta_{[1]}\Lambda_1\theta_{[1]}^{-1} \tag{4.9}$$

where $P_{[1]} = P$, $P_{[2]} = \tilde{P}$, $\theta_{[1]} = \theta_1 = \theta$, $\Lambda_1 = \Lambda$ and $\lambda = \lambda_1$. Then after n repeated Darboux transformations, we have

$$P_{[n+1]} = P_{[n]} - \theta_{[n]}\Lambda_n\theta_{[n]}^{-1} \tag{4.10}$$

in which $\theta_{[k]} = \phi_{[k]} \mid_{\phi\to\theta_k}$. We express $P_{[n+1]}$ in quasideterminant form as

$$P_{[n+1]} = P + \begin{vmatrix} \theta_1 & \theta_2 \;\dots\; \theta_n & 0_2 \\ \theta_1\Lambda_1 & \theta_2\Lambda_2 \dots \theta_n\Lambda_n & 0_2 \\ \vdots & \vdots \;\dots\; \vdots & \vdots \\ \theta_1\Lambda_1^{n-2} & \theta_2\Lambda_2^{n-2} \dots \theta_n\Lambda_n^{n-2} & 0_2 \\ \theta_1\Lambda_1^{n-1} & \theta_2\Lambda_2^{n-1} \dots \theta_n\Lambda_n^{n-1} & I_2 \\ \theta_1\Lambda_1^{n} & \theta_2\Lambda_2^{n} \dots \theta_n\Lambda_n^{n} & \boxed{0_2} \end{vmatrix}. \tag{4.11}$$

We now express each θ_i, Λ_i as a 2×2 matrix

$$\theta_i = \begin{pmatrix} \phi_{2i-1} & \phi_{2i} \\ \psi_{2i-1} & \psi_{2i} \end{pmatrix}, \; \Lambda_i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda_i \tag{4.12}$$

so that

$$\theta_i\Lambda_i^k = \begin{pmatrix} \phi_{2i-1} & (-1)^k\phi_{2i} \\ \psi_{2i-1} & (-1)^k\psi_{2i} \end{pmatrix} \lambda_i^k \tag{4.13}$$

for positive integers $i, k = 1, \dots, n$. Here the relation $\phi_{2i-1}\psi_{2i} + \phi_{2i}\psi_{2i-1} = 0$ holds.

Let

$$\Theta^{(n)} = (\theta_1\Lambda_1^n, \dots, \theta_n\Lambda_n^n) = \begin{pmatrix} \phi^{(n)} \\ \psi^{(n)} \end{pmatrix}, \tag{4.14}$$

where

$$\begin{aligned} \phi^{(n)} &= (\lambda_1^n\phi_1, (-\lambda_1)^n\,\phi_2, \dots, \lambda_n^n\phi_{2n-1}, (-\lambda_n)^n\,\phi_{2n}), \\ \psi^{(n)} &= (\lambda_1^n\psi_1, (-\lambda_1)^n\,\psi_2, \dots, \lambda_n^n\psi_{2n-1}, (-\lambda_n)^n\,\psi_{2n}) \end{aligned}$$

denote $1\times 2n$ row vectors. Thus, (4.11) can be rewritten as

$$P_{[n+1]} = P + \begin{vmatrix} \widehat{\Theta} & E \\ \theta^{(n)} & \boxed{0_2} \end{vmatrix}, \tag{4.15}$$

where $\widehat{\Theta} = \left(\theta_i\Lambda_i^{j-1}\right)_{i,j=1,\dots,n}$ and $E = (e_{2n-1}, e_{2n})$ denote $2n\times 2n$ and $2n\times 2$ matrices respectively, where e_i represents a column vector with 1 in the i^{th} row and zeros elsewhere. Hence, we obtain

$$P_{[n+1]} = P + \begin{pmatrix} \begin{vmatrix} \widehat{\Theta} & e_{2n-1} \\ \phi^{(n)} & \boxed{0} \end{vmatrix} & \begin{vmatrix} \widehat{\Theta} & e_{2n} \\ \phi^{(n)} & \boxed{0} \end{vmatrix} \\ \begin{vmatrix} \widehat{\Theta} & e_{2n-1} \\ \psi^{(n)} & \boxed{0} \end{vmatrix} & \begin{vmatrix} \widehat{\Theta} & e_{2n} \\ \psi^{(n)} & \boxed{0} \end{vmatrix} \end{pmatrix}. \tag{4.16}$$

By comparing with (4.6), we immediately see that $q_{[n+1]}$ and $r_{[n+1]}$ can be expressed as quasideterminants, namely,

$$q_{[n+1]} = q + 2i \begin{vmatrix} \widehat{\Theta} & e_{2n} \\ \phi^{(n)} & \boxed{0} \end{vmatrix}, \quad r_{[n+1]} = r - 2i \begin{vmatrix} \widehat{\Theta} & e_{2n-1} \\ \psi^{(n)} & \boxed{0} \end{vmatrix}. \tag{4.17}$$

We now consider the linear eigenvalue problems $L(\Phi_i) = M(\Phi_i) = 0$, where the operators L, M are given in (2.3)-(2.4) and Φ_i denotes n distinct eigenfunctions as

$$\Phi_i = \begin{pmatrix} \phi_i \\ \psi_i \end{pmatrix} (i = 1, \ldots, n). \tag{4.18}$$

Thus, the pair $q_{[n+1]}$ and $r_{[n+1]}$ are written with respect to n, where n is an odd $(n = 2k - 1)$ or even number $(n = 2k)$, and $k \in \mathbb{N}$ is a positive integer.

In the case of n odd $(n = 2k - 1)$

$$q_{[n+1]} = q + 2i \begin{vmatrix} \psi_1 & \psi_2 & \ldots & \psi_n & 0 \\ \phi_1\lambda_1 & \phi_2\lambda_2 & \ldots & \phi_n\lambda_n & 0 \\ \psi_1\lambda_1^2 & \psi_2\lambda_2^2 & \ldots & \psi_n\lambda_n^2 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \phi_1\lambda_1^{n-2} & \phi_2\lambda_2^{n-2} & \ldots & \phi_n\lambda_n^{n-2} & 0 \\ \psi_1\lambda_1^{n-1} & \psi_2\lambda_2^{n-1} & \ldots & \psi_n\lambda_n^{n-1} & 1 \\ \phi_1\lambda_1^n & \phi_2\lambda_2^n & \ldots & \phi_n\lambda_n^n & \boxed{0} \end{vmatrix}, \tag{4.19}$$

$$r_{[n+1]} = r - 2i \begin{vmatrix} \phi_1 & \phi_2 & \ldots & \phi_n & 0 \\ \psi_1\lambda_1 & \psi_2\lambda_2 & \ldots & \psi_n\lambda_n & 0 \\ \phi_1\lambda_1^2 & \phi_2\lambda_2^2 & \ldots & \phi_n\lambda_n^2 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \psi_1\lambda_1^{n-2} & \psi_2\lambda_2^{n-2} & \ldots & \psi_n\lambda_n^{n-2} & 0 \\ \phi_1\lambda_1^{n-1} & \phi_2\lambda_2^{n-1} & \ldots & \phi_n\lambda_n^{n-1} & 1 \\ \psi_1\lambda_1^n & \psi_2\lambda_2^n & \ldots & \psi_n\lambda_n^n & \boxed{0} \end{vmatrix}. \tag{4.20}$$

For $n = 1$, we obtain a pair of new solutions for the Gerdjikov-Ivanov system (2.1)-(2.2)

$$\begin{aligned} q_{[2]} &= q + 2i \begin{vmatrix} \psi_1 & 1 \\ \phi_1\lambda_1 & \boxed{0} \end{vmatrix} \\ &= q - 2i\lambda_1 \frac{\phi_1}{\psi_1}, \end{aligned} \tag{4.21}$$

$$\begin{aligned} r_{[2]} &= r - 2i \begin{vmatrix} \phi_1 & 1 \\ \psi_1\lambda_1 & \boxed{0} \end{vmatrix} \\ &= r + 2i\lambda_1 \frac{\psi_1}{\phi_1}, \end{aligned} \tag{4.22}$$

where $\Phi_1 = (\phi_1, \psi_1)^T$ is a solution of the eigenvalue problems $L(\Phi_1) = M(\Phi_1) = 0$.

In the case of n even ($n=2k$)

$$q_{[n+1]}=q+2i\begin{vmatrix} \phi_1 & \phi_2 & \dots & \phi_n & 0 \\ \psi_1\lambda_1 & \psi_2\lambda_2 & \dots & \psi_n\lambda_n & 0 \\ \phi_1\lambda_1^2 & \phi_2\lambda_2^2 & \dots & \phi_n\lambda_n^2 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \phi_1\lambda_1^{n-2} & \phi_2\lambda_2^{n-2} & \dots & \phi_n\lambda_n^{n-2} & 0 \\ \psi_1\lambda_1^{n-1} & \psi_2\lambda_2^{n-1} & \dots & \psi_n\lambda_n^{n-1} & 1 \\ \phi_1\lambda_1^{n} & \phi_2\lambda_2^{n} & \dots & \phi_n\lambda_n^{n} & \boxed{0} \end{vmatrix}, \tag{4.23}$$

$$r_{[n+1]}=r-2i\begin{vmatrix} \psi_1 & \psi_2 & \dots & \psi_n & 0 \\ \phi_1\lambda_1 & \phi_2\lambda_2 & \dots & \phi_n\lambda_n & 0 \\ \psi_1\lambda_1^2 & \psi_2\lambda_2^2 & \dots & \psi_n\lambda_n^2 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \psi_1\lambda_1^{n-2} & \psi_2\lambda_2^{n-2} & \dots & \psi_n\lambda_n^{n-2} & 0 \\ \phi_1\lambda_1^{n-1} & \phi_2\lambda_2^{n-1} & \dots & \phi_n\lambda_n^{n-1} & 1 \\ \psi_1\lambda_1^{n} & \psi_2\lambda_2^{n} & \dots & \psi_n\lambda_n^{n} & \boxed{0} \end{vmatrix}. \tag{4.24}$$

For $n=2$, we have

$$q_{[3]}=q+2i\begin{vmatrix} \phi_1 & \phi_2 & 0 \\ \psi_1\lambda_1 & \psi_2\lambda_2 & 1 \\ \phi_1\lambda_1^2 & \phi_2\lambda_2^2 & \boxed{0} \end{vmatrix}, \tag{4.25}$$

$$r_{[3]}=r-2i\begin{vmatrix} \psi_1 & \psi_2 & 0 \\ \phi_1\lambda_1 & \phi_2\lambda_2 & 1 \\ \psi_1\lambda_1^2 & \psi_2\lambda_2^2 & \boxed{0} \end{vmatrix}. \tag{4.26}$$

Thus, we obtain a pair of new solutions for the system (2.1)-(2.2), namely

$$q_{[3]} = q-2i\left(\lambda_1^2-\lambda_2^2\right)\frac{\phi_1\phi_2}{\lambda_1\psi_1\phi_2-\lambda_2\phi_1\psi_2}, \tag{4.27}$$

$$r_{[3]} = r+2i\left(\lambda_1^2-\lambda_2^2\right)\frac{\psi_1\psi_2}{\lambda_1\phi_1\psi_2-\lambda_2\psi_1\phi_2}, \tag{4.28}$$

where $\Phi_i=(\phi_i,\psi_i)^T$ is a solution of the eigenvalue problems $L(\Phi_i)=M(\Phi_i)=0\ \ (i=1,2)$.

Reduction

The eigenfunction $\Phi_k=\begin{pmatrix}\phi_k\\ \psi_k\end{pmatrix}$ associated with the eigenvalue λ_k has the following relations when we choose the reduction $r_{[n+1]}=q^*_{[n+1]}$:

$$\psi_k = \phi_k^* \text{ for real } \lambda_k, \tag{4.29}$$

$$\psi_k = \phi_l^* \text{ when } \lambda_k=\lambda_l^*\ (k\neq l), \tag{4.30}$$

where $k \in \mathbb{N}$. There are many ways to guarantee the reduction $r_{[n+1]} = q^*_{[n+1]}$ for the n-fold Darboux transformations when $n > 2$. In the present paper we will restrict ourselves to the reductions $r_{[2]} = q^*_{[2]}$ and $r_{[3]} = q^*_{[3]}$. For the one-fold Darboux transformation, the reduction $r_{[2]} = q^*_{[2]}$ implies

$$\psi_1 = \phi_1^* \ \text{ for } \ \lambda_1 \in \mathbb{R}. \tag{4.31}$$

Furthermore, for the two-fold Darboux transformation, in order that $r_{[3]} = q^*_{[3]}$, the eigenfunctions $\Phi_1 = (\phi_1, \psi_1)^T$ and $\Phi_2 = (\phi_2, \psi_2)^T$ with the eigenvalues λ_1, λ_2, either of the following conditions hold:

$$\psi_1 = \phi_1^*, \ \psi_2 = \phi_2^* \ \text{ for } \ \lambda_1, \lambda_2 \in \mathbb{R} \text{ or} \tag{4.32}$$

$$\psi_1 = \phi_2^*, \ \psi_2 = \phi_1^* \ \text{ for } \ \lambda_2^* = \lambda_1. \tag{4.33}$$

5 Particular solutions

Let us consider the spectral problem $L(\Phi) = M(\Phi) = 0$ with eigenvalue λ, where $\Phi = (\phi, \psi)^T$ and L, M are given by (2.3)-(2.4) so that

$$\Phi_x + J\Phi\lambda^2 - R\Phi\lambda + \frac{1}{2}qrJ\Phi = 0, \tag{5.1}$$

$$\Phi_t + 2J\Phi\lambda^4 - 2R\Phi\lambda^3 + qrJ\Phi\lambda^2 + U\Phi\lambda + W\Phi = 0. \tag{5.2}$$

5.1 Solutions for the vacuum

For $q = r = 0$, the above equations transform into the first-order linear system

$$\Phi_x + J\Phi\lambda^2 = 0 \tag{5.3}$$

$$\Phi_t + 2J\Phi\lambda^4 = 0 \tag{5.4}$$

which has solution

$$\phi_k = e^{-i\lambda_k^2\left(x+2\lambda_k^2 t\right)}, \ \ \psi_k = e^{i\lambda_k^2\left(x+2\lambda_k^2 t\right)}, \tag{5.5}$$

where $k \in \mathbb{N}$.

Case 1 ($n = 1$)

For one single Darboux transformation, due to the required reduction $r = q^*$, we must take λ_1 to be real and $\psi_1 = \phi_1^*$. By substituting $\phi_1 = e^{-i\lambda_1^2\left(x+2\lambda_1^2 t\right)}$ and $\psi_1 = e^{i\lambda_1^2\left(x+2\lambda_1^2 t\right)}$ into (4.21), we obtain a new solution $q_{[2]}$ for the GI equation (1.1) as

$$q_{[2]} = -2i\lambda_1 e^{-2i\lambda_1^2\left(x+2\lambda_1^2 t\right)}, \tag{5.6}$$

where $r_{[2]} = q^*_{[2]}$. This, of course, is not a soliton but a periodic solution. It is obvious that $|q_{[2]}|^2 = \textit{constant}$ so that it satisfies a linear equation $iq_t + q_{xx} = 0$ obtained from (1.1). However, this is not an interesting solution obtained by the use of the Darboux transformation.

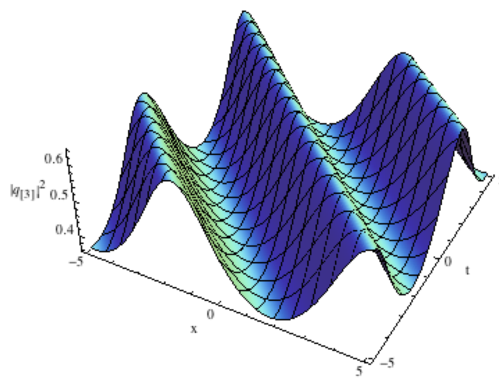

FIGURE 1. Periodic solution $|q_{[3]}|^2$ of the GI equation (1.1) with $\lambda_1 = 0.1, \lambda_2 = 0.7$.

Case 2 $(n = 2)$

In order that $r_{[3]} = q^*_{[3]}$, λ_1 and λ_2 are either real or complex conjugate eigenvalues to each other.

Case 2a

Under the condition (4.32), (4.27) yields a periodic solution

$$q_{[3]} = -2i \frac{\lambda_1^2 - \lambda_2^2}{\lambda_1 e^{2i\lambda_1^2\left(x+2\lambda_1^2 t\right)} - \lambda_2 e^{2i\lambda_2^2\left(x+2\lambda_2^2 t\right)}} \tag{5.7}$$

which can be rewritten as

$$\left|q_{[3]}\right|^2 = 4 \frac{\left(\lambda_1^2 - \lambda_2^2\right)^2}{\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2 \cos\gamma}, \tag{5.8}$$

where $\gamma = 2\left(\lambda_1^2 - \lambda_2^2\right)\left[x + 2(\lambda_1^2 + \lambda_2^2)t\right]$. Here, it can be easily seen that the denominator of the function $\left|q_{[3]}\right|^2$ is positive since $0 < (\lambda_1 - \lambda_2)^2 \leq \lambda_1^2 + \lambda_2^2 - \lambda_1\lambda_2 \cos\gamma \leq (\lambda_1 + \lambda_2)^2$. The solution (5.8) is plotted in the figure 1.

Case 2b

For the choice (4.33), (4.27) leads to

$$q_{[3]} = -2i \frac{\lambda_1^2 - \lambda_2^2}{\lambda_1 e^{2i\lambda_1^2\left(x+2\lambda_1^2 t\right)} - \lambda_2 e^{2i\lambda_2^2\left(x+2\lambda_2^2 t\right)}}. \tag{5.9}$$

By taking $\lambda_1 = \xi + i\eta$ and $\lambda_2 = \xi - i\eta$, we obtain one soliton solution of the GI equation (1.1) as

$$\left|q_{[3]}\right|^2 = 32 \, \frac{\xi^2\eta^2}{\eta^2 - \xi^2 + (\xi^2 + \eta^2)\cosh\left(8\xi\eta\left[x + 4\left(\xi^2 - \eta^2\right)\right]\right)}, \tag{5.10}$$

where $\xi, \eta \in \mathbb{R}$. The figure 2 demonstrates one soliton solution.

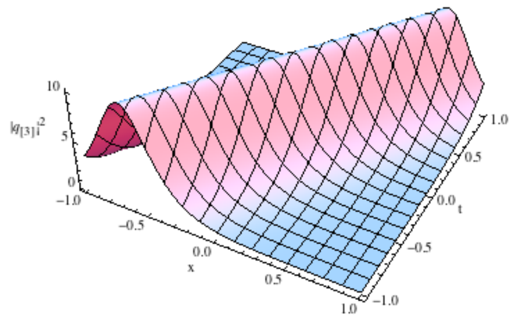

FIGURE 2. One soliton solution $|q_{[3]}|^2$ of the GI equation (1.1) when $\xi = 0.8, \eta = 0.9$.

5.2 Solutions for non-zero seeds

For $q, r \neq 0$ and $r = q^*$, it is easily seen that

$$q = ke^{i\left[ax+\left(ak^2+\frac{1}{2}k^4-a^2\right)t\right]} \tag{5.11}$$

is a periodic solution of the GI equation (1.1), where a and k are real numbers. We use this as the seed solution for application of Darboux transformations.

Substituting (5.11) into the linear system (5.1)-(5.2) and then solving for the eigenfunction $\Phi = (\phi, \psi)^T$, we obtain

$$\phi(x,t,\lambda) = c_1 e^{\frac{i}{2}\left([a+D]x+\left[b-\left(a-2\lambda^2\right)D\right]t\right)} + c_2 e^{\frac{i}{2}\left([a-D]x+\left[b+\left(a-2\lambda^2\right)D\right]t\right)}, \tag{5.12}$$

$$\psi(x,t,\lambda) = \widetilde{c_1} e^{-\frac{i}{2}\left([a+D]x+\left[b-\left(a-2\lambda^2\right)D\right]t\right)} + \widetilde{c_2} e^{-\frac{i}{2}\left([a-D]x+\left[b+\left(a-2\lambda^2\right)D\right]t\right)}, \tag{5.13}$$

where $b = ak^2 - a^2 + \frac{k^4}{2}$, $D = \sqrt{a^2 + 4a\lambda^2 + 4\lambda^4 + k^4 + 2ak^2}$, $\widetilde{c_1} = i\left(\frac{k^2+a+2\lambda^2-D}{2k\lambda}\right)c_2$, $\widetilde{c_2} = i\left(\frac{k^2+a+2\lambda^2+D}{2k\lambda}\right)c_1$ and c_1, c_2 are integration constants, obtained from (5.1)-(5.2).

Case 3 ($n = 1$)

For the one-fold Darboux transformation, it can easily be shown that $D^2(\lambda_1) > 0$ and $D^2(\lambda_1) < 0$ produce the periodic and soliton solutions respectively of the GI equation. For example, for $D^2(\lambda_1) > 0$ with $k^2 = -2a$, (4.21) yields a periodic solution

$$\left|q_{[2]}\right|^2 = 2\frac{\left(a+2\lambda_1^2\right)^2}{2\lambda_1^2 - a - 2k\lambda_1 \sin\gamma}, \tag{5.14}$$

where $\gamma = \left(a + 2\lambda_1^2\right)x + (4\lambda_1^4 - a^2)t$. In this solution, it should be observed that the denominator must be positive since $2\lambda_1^2 - a - 2k\lambda_1 \sin\gamma = \frac{1}{2}\left(4\lambda_1^2 + k^2 - 4k\lambda_1 \sin\gamma\right) \geq \frac{1}{2}\left(2\lambda_1 - k\right)^2 > 0$ for $k \neq 2\lambda_1$. The solution (5.14) is plotted in the figure 3.

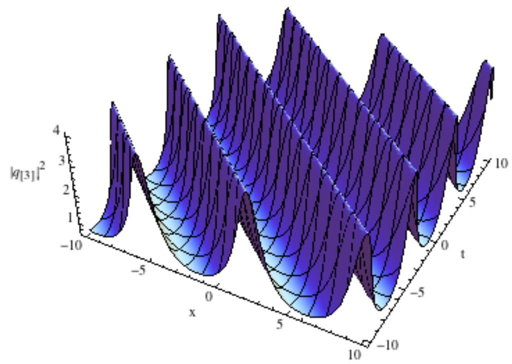

FIGURE 3. Periodic solution $|q_{[2]}|^2$ of the GI equation (1.1) with the choice of parameters $\lambda_1 = 0.3, k = \sqrt{2}$.

Case 4 ($n = 2$)

In this case, we have two eigenvalues λ_1 and λ_2. For solutions such that $r_{[3]} = q^*_{[3]}$, these eigenvalues are either real or complex conjugate to each other. The functions $\phi_i, \psi_i (i = 1, 2)$ with the eigenvalues λ_1, λ_2 either hold $(R1)$ $\psi_1 = \phi_1^*, \psi_2 = \phi_2^*$ for $\lambda_1, \lambda_2 \in \mathbb{R}$ or $(R2)$ $\psi_1 = \phi_2^*, \psi_2 = \phi_1^*$ for $\lambda_1 = \lambda_2^*$. An example for $(R2)$ is given below.

The solution (4.27) under the choice $(R2)$ can be rewritten as

$$q_{[3]} = q + 2i\Lambda \frac{\phi_1\phi_2}{\lambda_2 |\phi_1|^2 - \lambda_1 |\phi_2|^2}, \tag{5.15}$$

where $\Lambda = \lambda_1^2 - \lambda_2^2$ such that $\Lambda \in i\mathbb{R}$. For simplicity, let us choose $k^2 = -2a$, then we find

$$q_{[3]} = \frac{k\lambda_2\Lambda_2 e^{i\left([a+\Lambda]x - \left[a^2 - 2\kappa\Lambda\right]t\right)} - k\lambda_1\Lambda_1 e^{i\left([a-\Lambda]x - \left[a^2+2\kappa\Lambda\right]t\right)} - 4i\lambda_1\lambda_2\Lambda e^{-i\left(\kappa x + 2\left[\lambda_1^4 + \lambda_2^4\right]t\right)}}{\lambda_2\Lambda_1 e^{i\Lambda(x+2\kappa t)} - \lambda_1\Lambda_2 e^{-i\Lambda(x+2\kappa t)} - ik\Lambda e^{i\left([a+\kappa]x - \left[a^2 - 2\lambda_1^4 - 2\lambda_2^4\right]t\right)}}, \tag{5.16}$$

where $\Lambda_1 = a + 2\lambda_1^2$, $\Lambda_2 = a + 2\lambda_2^2$ and $\kappa = \lambda_1^2 + \lambda_2^2$ such that $\kappa \in \mathbb{R}$.

Let $\lambda_1 = \xi + i\eta$ and $\lambda_2 = \xi - i\eta$, where $\xi, \eta \neq 0$. Then

$$\left|q_{[3]}\right|^2 = k^2 + 16\xi\eta \; \frac{m_0 + m_1 \cos\gamma_1 \sinh\gamma_2 + m_2 \sin\gamma_1 \cosh\gamma_2}{n_0 + n_1 \cosh(2\gamma_2) + n_2 \cos\gamma_1 \sinh\gamma_2 + n_3 \sin\gamma_1 \cosh\gamma_2}, \tag{5.17}$$

where

$$\begin{aligned}
\gamma_1 &= \left(a + 2\left[\xi^2 - \eta^2\right]\right)x - \left(a^2 - 4\left[\xi^2 - \eta^2\right]^2 + 16\xi^2\eta^2\right)t, \\
\gamma_2 &= 4\xi\eta\left(x + 4\left[\xi^2 - \eta^2\right]t\right), \\
m_0 &= 2\xi\eta\left(3a^2 + 4a\left[\xi^2 - \eta^2\right] - 4\left[\xi^2 + \eta^2\right]^2\right), \\
m_1 &= k\xi\left(a^2 + 4\left[\xi^2 + \eta^2\right]\left[a + \xi^2 - 3\eta^2\right]\right), \\
m_2 &= k\xi\left(a^2 - 4\left[\xi^2 + \eta^2\right]\left[a + 3\xi^2 - \eta^2\right]\right), \\
n_0 &= \left(\xi^2 - \eta^2\right)\left(a + 2\xi^2 + 2\eta^2\right)^2 + 8a\eta^2\left(3\xi^2 + \eta^2\right),
\end{aligned}$$

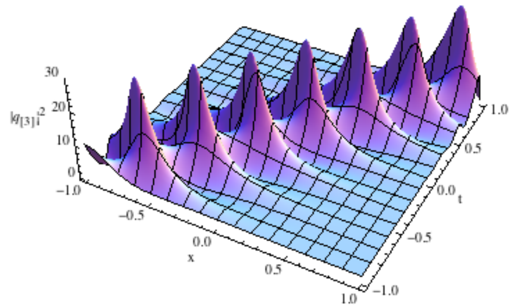

FIGURE 4. Breather solution $|q_{[3]}|^2$ of the GI equation (1.1) with the parameters chosen as $\xi = 1, \eta = 1.1, k = \sqrt{2}$.

$$
\begin{aligned}
n_1 &= -\left(\xi^2+\eta^2\right)\left(\left[a+2\xi^2+2\eta^2\right]^2-8a\eta^2\right), \\
n_2 &= 8k\xi^2\eta\left(a+2\xi^2+2\eta^2\right), \\
n_3 &= 8k\xi^2\eta^2\left(a-2\xi^2-2\eta^2\right).
\end{aligned}
$$

By choosing appropriate parameters, the breather solution of the Gerdijikov-Ivanov equation (1.1) is plotted in the figure 4. Similarly, for the choice ($R1$), (4.27) gives us a periodic solution.

6 Conclusion

In this paper, we have presented a standard Darboux transformation for the GI equation (1.1). We have constructed solutions in quasideterminant forms for the GI equation. These quasideterminants are expressed in terms of solutions of the linear partial differential equations given by (5.1)-(5.2). These solutions arise naturally from the Darboux transformation we present here. Moreover, parametric, soliton and breather solutions for zero and non-zero seeds have been given as particular examples for the GI equation. Examples of these particular solutions are plotted in the figures 1–4 with the chosen parameters. It should be emphasised that we may derive several types of particular solutions for the GI equation by using the Darboux transformation we present here. Finally, it should be pointed out that the Darboux transformation technique is a universal instrument that allows us to construct exact solutions for other integrable systems.

Acknowledgments

The author would like to thank Dr. Jonathan Nimmo for his valuable comments and suggestions.

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